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Some properties of Poisson-type equations

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Abstract. The general equation $\nabla^2\Phi = -4\pi f$ is studied where $f \neq 0$ all over a three-dimensional Euclidean space and has no symmetries. The necessary and sufficient conditions for the existence of a solution vanishing at infinity are derived. Some specific differential equations of the above form appearing in general relativity are explicitly solved. Several properties of the solutions are established.

1. Introduction

The solution of Poisson's equation

$$\nabla^2\Phi = -4\pi\rho(\boldsymbol{r}) \tag{1}$$

in a three-dimensional Euclidean space, where $\rho(\boldsymbol{r})$ is zero outside a sphere of finite radius, is a well known problem treated extensively in many advanced textbooks (Jackson 1962, Kellog 1953, MacMillan 1958, Morse and Feshbach 1953). In most of these presentations little attention is given to the more general Poisson-type equation

$$\nabla^2\Phi = -4\pi f(\boldsymbol{r}) \tag{2}$$

where now $f(\boldsymbol{r})$ is different from zero over all space. In fact only special cases of equation (2) are studied, e.g. with symmetries or of two-dimensional nature, where the problem can be solved with well known methods (e.g. complex variables).

This attitude in the literature is justified because of the problems encountered so far and the difficulty of solving equation (2) in the general case. Several investigations, however (Chandrasekhar and Esposito 1970, Persides 1971a, Winicour 1984), have shown that many approximation schemes for solving Einstein's equations in general relativity for an isolated source of spatially compact support reduce to a sequence of Poisson-type equations of the form

$$\nabla^2\Phi^{(n)} = -4\pi f(\boldsymbol{r})^{(n)} \tag{3}$$

where $f(\boldsymbol{r})^{(n)}$ is given explicitly in terms of $\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(n-1)}$. The 'superpotentials' $\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(n-1)}$ are known at the nth step and give $f(\boldsymbol{r})^{(n)} \neq 0$ over all space. Such is the case even in classical electrodynamics (Persides 1971b), if Maxwell's equations are solved by an expansion in powers of c^{-1} (a method equivalent to the use of the retarded potential). All these suggest to us to consider the general problem of solving equation (2) in a three-dimensional Euclidean space. It should be emphasised that all problems leading to equation (2) concern the field generated by an isolated bounded source. This is the real source. The field, however, of a certain approximation step acts as 'source' for the subsequent approximations and we have to solve equation (2)

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or (3) instead of equation (1). Since the real source is bounded we are interested in solutions of equation (2) which exhibit the same behaviour as the solutions of equation (1) at infinity, that is $\Phi(\boldsymbol{r})$ goes to zero as $r \to \infty$. In fact, in order that the calculations give some concrete and useful results we will examine solutions for which (a) $\Phi(\boldsymbol{r})$ is analytic in r^{-1} outside some sphere of finite radius and (b) $\Phi(\boldsymbol{r})$ presents no singularities of any kind for finite r.

The purpose of this paper is to examine the general relations between $\Phi(\boldsymbol{r})$ and $f(\boldsymbol{r})$ in the case when equation (2) can be solved and to give explicit solutions for some Poisson-type equations.

In § 2 we examine the conditions on $f(\boldsymbol{r})$ which guarantee the existence of a solution Φ analytic in r^{-1} and present some properties of the solutions. In § 3 we consider sequences of specific Poisson-type equations arising in the study of gravitational radiation from isolated sources. The solutions of these equations can eventually be expressed as integrals over a finite region of space (that of the real source) although $f(\boldsymbol{r})$ is different from zero over all space. Finally in § 4 we close with some remarks concerning more general Poisson-type equations.

Throughout this paper $\rho(\boldsymbol{r})$ and $f(\boldsymbol{r})$ are assumed to be continuous and bounded functions of $\boldsymbol{r}$ such that the volume integrals $\int \rho(\boldsymbol{r}')|\boldsymbol{r}-\boldsymbol{r}'|^{-1}\,\mathrm{d}V'$ and $\int f(\boldsymbol{r}')|\boldsymbol{r}-\boldsymbol{r}'|^{-1}\,\mathrm{d}V'$ over any sphere of finite radius exist. Furthermore, it is assumed that for $\rho(\boldsymbol{r})$ an r_0 exists such that $\rho(\boldsymbol{r})=0$ for $r>r_0$, while $f(\boldsymbol{r})$ may be different from zero for arbitrarily large r. In the summations used below, the indices n, l, m are assumed to take values from 0 to ∞, from 0 to ∞ and from $-l$ to l respectively, unless otherwise noted. Finally in volume integrals the integration is assumed to be carried out over that part (finite or infinite) of space in which the integrand is different from zero.

2. Existence and properties of solutions

It is well known that the unique solution of equation (1) that goes to zero as $r \to \infty$ is

$$\Phi(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'. \tag{4}$$

Furthermore for large enough r ($r>r_0$) we have

$$\Phi(\boldsymbol{r}) = \sum_n \Phi_n r^{-n-1}. \tag{5}$$

Thus the first objective is to ask for the necessary and sufficient conditions that permit a solution of equation (2) to be written in the form (5). The following theorem provides an answer.

Theorem. Let $f(\boldsymbol{r})$ be a continuous and bounded function of $\boldsymbol{r}$. The necessary and sufficient conditions for equation (2) to have a solution of the form (5) for $r>r_0$ are

$$f(\boldsymbol{r}) = \sum_n f_n r^{-n-3}, \tag{6}$$

$$\int f_n(\theta, \varphi)\, Y_{nm}(\theta, \varphi)\,\mathrm{d}\Omega = 0, \tag{7}$$

where $Y_{lm}(\theta, \varphi)$ are the usual spherical harmonics and there is no summation over n in equation (7).

Proof. We establish first the necessity of conditions (6) and (7). Let Φ be a solution of equation (2) and have the form (5) for $r > r_0$. Acting on Φ with the Lagrangian operator ∇^2 expressed in spherical coordinates r, θ, φ we find

$$\nabla^2\Phi = \sum_n \left[n(n+1)\Phi_n + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi_n}{\partial\varphi^2}\right] r^{-n-3}. \tag{8}$$

Since Φ_n is a function of θ and φ we can write

$$\Phi_n = \sum_{l,m} A_{nlm} Y_{lm}(\theta, \varphi) \tag{9}$$

where A_{nlm} are constants. Substituting in (8) and using well known properties of spherical harmonics we find

$$\nabla^2\Phi = \sum_{n,l,m} A_{nlm}[n(n+1) - l(l+1)] Y_{lm} r^{-n-3}. \tag{10}$$

Hence the right-hand side of (2) has the form (6) with

$$f_n = -(4\pi)^{-1} \sum_{l,m} (n-l)(n+l+1) A_{nlm} Y_{lm}(\theta, \varphi). \tag{11}$$

We multiply (11) by $Y^*_{nm'}$ and integrate over θ and φ. Using the orthogonality property of spherical harmonics we have

$$-4\pi \int f_n Y^*_{nm'}\, \mathrm{d}\Omega = \sum_{l,m} (n-l)(n+l+1) A_{nlm} \delta_{ln} \delta_{mm'} = 0 \tag{12}$$

which implies equation (7).

Let us now assume that equations (6) and (7) hold for $r > r_0$. Then a solution of (2) is

$$\Phi(\boldsymbol{r}) = \int \frac{f(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\, \mathrm{d}V'. \tag{13}$$

We will show that this solution can be written in the form (5) for $r > r_0$. After that it follows that this is the only solution which goes to zero at infinity (if there was another Φ', the difference $\Phi - \Phi'$ would satisfy the Laplace equation everywhere and thus it would be identically zero).

For $r > r_0$ the solution (13) can be split into three integrals I_1, I_2, I_3 as follows:

$$\Phi = \int_{0<r'<r_0} \frac{f(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, \mathrm{d}V' + \int_{r_0<r'<r} \frac{f(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, \mathrm{d}V' + \int_{r<r'<\infty} \frac{f(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, \mathrm{d}V'. \tag{14}$$

We use the expansion (Jackson 1962, p 69)

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi). \tag{15}$$

The first integral of (14) is

$$I_1 = \sum_{l,m} \frac{4\pi}{(2l+1)r^{l+1}} Y_{lm}(\theta, \varphi) \int Y^*_{lm}(\theta', \varphi') \left(\int_0^{r_0} f(\boldsymbol{r}') r'^{l+2}\, \mathrm{d}r' \right) \mathrm{d}\Omega'.$$

The second integral of (14) is

$$I_2 = \int_{r_0<r'<r} \left(\sum_n f_n r'^{-n-3} \right) \left(\sum_{l,m} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi) \right) \mathrm{d}V'$$

$$= \sum_{n,l,m} \frac{4\pi}{(2l+1)r^{l+1}} Y_{lm}(\theta, \varphi) \int_{r_0}^{r} r'^{l-n-3} r'^2\, \mathrm{d}r' \int f_n(\theta', \varphi') Y^*_{lm}(\theta', \varphi')\, \mathrm{d}\Omega'$$

$$= \sum_{\substack{n,l,m \\ n\neq l}} \frac{4\pi}{(2l+1)(l-n)} \left(\frac{1}{r^{n+1}} - \frac{r_0^{l-n}}{r^{l+1}} \right) Y(\theta, \varphi) \int f_n(\theta', \varphi') Y^*_{lm}(\theta', \varphi') \, d\Omega'.$$

The third integral of (14) is

$$I_3 = \int_{r<r'<\infty} \left(\sum_n f_n r'^{-n-3} \right) \left(\sum_{l,m} \frac{4\pi}{2l+1} \frac{r^l}{r'^{l+1}} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi) \right) dV'$$

$$= \sum_{n,l,m} \frac{4\pi r^l}{2l+1} Y_{lm}(\theta, \varphi) \int_r^\infty r'^{-l-n-4} r'^2 \, dr' \int f_n(\theta', \varphi') Y^*_{lm}(\theta', \varphi') \, d\Omega'$$

$$= \sum_{n,l,m} \frac{4\pi}{(2l+1)(l+n+1) r^{n+1}} Y_{lm}(\theta, \varphi) \int f_n(\theta', \varphi') Y^*_{lm}(\theta', \varphi') \, d\Omega'.$$

Summing up, we find from equation (14)

$$\Phi = \sum_l \frac{1}{r^{l+1}} \sum_m \frac{4\pi}{2l+1} Y_{lm}(\theta, \varphi) \int Y^*_{lm}(\theta', \varphi') \left(\int_0^{r_0} f(\mathbf{r}') r'^{l+2} \, dr' + \sum_{\substack{n \\ n\neq l}} \frac{r_0^{l-n}}{n-l} f_n(\theta', \varphi') \right) d\Omega'$$

$$+ \sum_n \frac{1}{r^{n+1}} \sum_{\substack{l,m \\ l\neq n}} \frac{4\pi}{(l-n)(l+n+1)} Y_{lm}(\theta, \varphi) \int f_n(\theta', \varphi') Y^*_{lm}(\theta', \varphi') \, d\Omega'. \tag{16}$$

Thus equation (5) has been proven. It should be emphasised that the quantity in large round brackets in (16) does not depend on r_0 as can be easily shown by direct differentiation and use of equations (6) and (7). This completes the proof of the theorem.

The details of the proof show that condition (6) guarantees the convergence of the integral I_3, while condition (7) eliminates terms of the form $r^{-n} \ln r (n \geq 1)$ from the potential (5). If condition (7) is not included in the theorem then Φ will behave as $(\ln r)/r$ near $r = \infty$. This behaviour is physically unacceptable since $\lim_{r\to\infty}(r\Phi)$ is essentially the radiated energy in gravitational radiation and has to be finite. More generally it can be argued that a physical field F (e.g. an electromagnetic field or a gravitational field expressed as the difference between the metric tensor and the flat metric background) has to fall off at least as fast as r^{-1} for large r since rF is the field registered on the conformal boundary of spacetime in the Penrose sense.

A useful property of the potential can be derived directly from the proof of the theorem. From equation (16) the coefficient of r^{-1} is

$$\Phi_0 = 4\pi Y_{00}(\theta, \varphi) \int Y^*_{00}(\theta', \varphi') \left(\int_0^{r_0} f(\mathbf{r}') r'^2 \, dr' + \sum_{n=1}^\infty \frac{r_0^{-n}}{n} f_n(\theta', \varphi') \right) d\Omega'$$

$$+ \sum_{l=1,m}^\infty \frac{4\pi}{l(l+1)} Y_{lm}(\theta, \varphi) \int f_0(\theta', \varphi') Y^*_{lm}(\theta', \varphi') \, d\Omega'$$

$$= \int_{\substack{\text{all} \\ \text{space}}} f(\mathbf{r}') \, dV' + \sum_{l=1,m} \frac{4\pi}{l(l+1)} Y_{lm}(\theta, \varphi) \int f_0(\theta', \varphi') Y^*_{lm}(\theta', \varphi') \, d\Omega'. \tag{17}$$

This expression gives the most important term (the coefficient of r^{-1}) in the expansion of the potential. The first term in (17) does not depend on θ and φ. The remaining terms included in the sum of (17) depend on θ and φ if and only if f_0 depends on θ and φ. Thus Φ_0 is independent of θ and φ and is equal to

$$\Phi_0 = \int_{\substack{\text{all}\\ \text{space}}} f(\boldsymbol{r}')\,\mathrm{d}V' \tag{18}$$

if and only if f_0 is independent of θ and φ. Note that f_0 cannot be a non-zero constant because then equation (7) for $n = m = 0$ gives $f_0 = 0$. Thus f_0 either depends on θ and φ in which case the full equation (17) holds or it is zero in which case equation (18) holds.

3. Sequences of Poisson-type equations

We turn our attention now to the solution of equation (3). If $f(\boldsymbol{r})^{(n)}$ is a non-linear complicated function of $\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(n-1)}$, then a special study is needed to improve our knowledge beyond that furnished by the theorem of § 2. It happens, however, that in some cases having to do with (electromagnetic or gravitational) radiation from a bounded source (Persides 1971b, 1985) the right-hand side of equation (3) can be written as a sum $f_1^{(n)} + f_2^{(n)} + f_3^{(n)}$ of simpler terms. Thus $f_1^{(n)}$ is different from zero in a bounded region of space (as ρ), $f_2^{(n)}$ is of the form $\hat{\boldsymbol{r}} \cdot \nabla\Phi^{(n)} + r^{-1}\Phi^{(n)}$ and $f_3^{(n)}$ is some complicated expression (usually highly non-linear). Thus the first problem is to solve the Poisson-type equation

$$\nabla^2\Psi = 2(\hat{\boldsymbol{r}} \cdot \nabla\Phi + r^{-1}\Phi) \tag{19}$$

where Φ satisfies equation (1) and $\hat{\boldsymbol{r}}$ represents the unit vector along $\boldsymbol{r}$. Since this equation is of the form (2), namely the right-hand side is non-zero over all space, it appears that the solution Ψ cannot be expressed as an integral over a finite region of space. However, a lengthy analysis of the right-hand side of (19) indicates that a solution can be written in the form

$$\Psi = \int \rho(\boldsymbol{r}')\,\frac{r - r' - |\boldsymbol{r} - \boldsymbol{r}'|}{|\boldsymbol{r} - \boldsymbol{r}'|}\,\mathrm{d}V'. \tag{20}$$

Having this expression we can now give a straightforward proof that this Ψ satisfies (19). The proof makes use of the relations

$$\nabla\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{\boldsymbol{r}' - \boldsymbol{r}}{|\boldsymbol{r} - \boldsymbol{r}'|^3}, \tag{21}$$

$$\nabla^2\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{22}$$

$$\nabla(r - |\boldsymbol{r} - \boldsymbol{r}'|)^n = n(r - |\boldsymbol{r} - \boldsymbol{r}'|)^{n-1}\left(\hat{\boldsymbol{r}} + \frac{\boldsymbol{r}' - \boldsymbol{r}}{|\boldsymbol{r} - \boldsymbol{r}'|}\right), \tag{23}$$

$$\begin{aligned}\nabla^2(r - |\boldsymbol{r} - \boldsymbol{r}'|)^n &= 2n(n-1)(r - |\boldsymbol{r} - \boldsymbol{r}'|)^{n-2}\left(1 + \frac{\hat{\boldsymbol{r}} \cdot \boldsymbol{r}' - r}{|\boldsymbol{r} - \boldsymbol{r}'|}\right)\\ &\quad + 2n(r - |\boldsymbol{r} - \boldsymbol{r}'|)^{n-1}\left(\frac{1}{r} - \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right),\end{aligned} \tag{24}$$

for $n=1$. Taking the Laplacian of equation (20) and forming the right-hand side of equation (19) from equation (4), we find identical expressions. Consequently, if Φ satisfies equation (1) the solution of (19) which goes to zero at infinity is given by (20). The integration in (20) is over that part of space where $\rho(\boldsymbol{r})\neq 0$, namely a finite region.

The above procedure can be generalised to provide us with explicit solutions of sequences of Poisson-type equations. Let the superpotentials $\Phi^{(n)}$, $n=0,1,\ldots,\infty$, satisfy the sequence of differential equations

$$\nabla^2\Phi^{(n)} = -(4\pi r^n/n!)\rho(\boldsymbol{r}) + 2(\hat{\boldsymbol{r}}\cdot\nabla + r^{-1})\Phi^{(n-1)} \tag{25}$$

and go to zero as $r\to\infty$ (we assume $\Phi^{(-1)}\equiv 0$). Then they are given by

$$\Phi^{(n)} = \frac{1}{n!}\int \rho(\boldsymbol{r}')\frac{(r-|\boldsymbol{r}-\boldsymbol{r}'|)^n}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V' \tag{26}$$

where the integral is taken over the region of space where $\rho(\boldsymbol{r})\neq 0$ (a finite region). The proof is again straightforward although a little more cumbersome. Using (21)-(24) we have the Laplacian of $\Phi^{(n)}$,

$$\begin{aligned}
\nabla^2\Phi^{(n)} &= \frac{1}{n!}\int \rho(\boldsymbol{r}')\nabla^2\frac{(r-|\boldsymbol{r}-\boldsymbol{r}'|)^n}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V' \\
&= \frac{1}{n!}\int \rho(\boldsymbol{r}')\bigg[(r-|\boldsymbol{r}-\boldsymbol{r}'|)^n\nabla^2\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla^2(r-|\boldsymbol{r}-\boldsymbol{r}'|)^n \\
&\quad + 2\nabla(r-|\boldsymbol{r}-\boldsymbol{r}'|)^n\cdot\nabla\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\bigg]\,\mathrm{d}V' \\
&= -\frac{4\pi}{n!}r^n\rho(\boldsymbol{r}) + \frac{2}{(n-1)!}\int\rho(\boldsymbol{r}')\bigg[(r-|\boldsymbol{r}-\boldsymbol{r}'|)^{n-1}\frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'-r}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \\
&\quad + (r-|\boldsymbol{r}-\boldsymbol{r}'|)^{n-1}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|^2} + \frac{(n-1)(r-|\boldsymbol{r}-\boldsymbol{r}'|)^{n-2}}{|\boldsymbol{r}-\boldsymbol{r}'|}\left(1+\frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'-r}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) \\
&\quad + \frac{(r-|\boldsymbol{r}-\boldsymbol{r}'|)^{n-1}}{r|\boldsymbol{r}-\boldsymbol{r}'|} - \frac{(r-|\boldsymbol{r}-\boldsymbol{r}'|)^{n-1}}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\bigg]\,\mathrm{d}V'.
\end{aligned}$$

We express also the right-hand side of (25) using (21)-(24). We find the same expression as above. Thus we have proved that the superpotentials $\Phi^{(n)}$ as given by (26) satisfy (25).

Finally let us consider the superpotentials $\Psi^{(n)}$ satisfying the sequence of differential equations

$$\nabla^2\Psi^{(n)} = 2(\hat{\boldsymbol{r}}\cdot\nabla + r^{-1})\Phi^{(n)} \tag{27}$$

where $\Phi^{(n)}$ is given by (26). Working along the same lines we can show that these superpotentials are

$$\Psi^{(n)} = \frac{1}{(n+1)!}\int\rho(\boldsymbol{r}')\frac{(r-|\boldsymbol{r}-\boldsymbol{r}'|)^{n+1}-r'^{n+1}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'. \tag{28}$$

Note again that the integration is carried out over a finite region of space.

Before closing this section it is instructive to justify the usefulness of these superpotentials by describing briefly (and loosely) how they are involved in the study of gravitational radiation. In general relativity it turns out that the energy radiated by the gravitational field depends on the time derivative of the coefficient of r^{-1} of the metric tensor $g_{\mu\nu}$. The metric tensor can be expressed in terms of the superpotentials presented here. Thus if

$$\Phi^{(n)} = \Phi_0^{(n)} r^{-1} + O(r^{-2}) \tag{29}$$

the time derivative of $\Phi_0^{(n)}$ enters into the expression of the radiated energy. Since there are no monopole and dipole contributions the first $\Phi_0^{(n)}$ that contributes is $\Phi_0^{(2)}$. Since for $r > r'$

$$(r - |\boldsymbol{r} - \boldsymbol{r}'|)^n = (\hat{\boldsymbol{r}} \cdot \boldsymbol{r}')^n + O(r^{-1}) \tag{30}$$

we have from (26)

$$\Phi_0^{(2)} = \tfrac{1}{2} \int \rho(\boldsymbol{r}')(\hat{\boldsymbol{r}} \cdot \boldsymbol{r}')^2 \, \mathrm{d}V'. \tag{31}$$

This integral can be very easily expressed in terms of the quadrupole moment. Thus we conclude that the dominant term in gravitational radiation is quadrupole radiation and we can give an explicit expression of it. A detailed account of the solution of Einstein's equations using the results of this paper will be given elsewhere (Persides 1985).

4. General remarks

In the previous sections we have presented some Poisson-type equations, i.e. generalisations of the well known equation (1). We have established a theorem for the existence of solutions with the appropriate behaviour at infinity and we have solved some specific equations suggested by current research in general relativity. However, it should be emphasised that the equations studied here are the simplest generalisations of equation (1). Usually in general relativity equations of the form (3) are encountered where $f(\boldsymbol{r})^{(n)}$ depends not only on $\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(n-1)}$, but also on $\Phi^{(n)}$. Such an example is furnished by the equation

$$\nabla^2 A_{\alpha\beta} = \partial\Theta_\alpha / \partial x^\beta + \partial\Theta_\beta / \partial x^\alpha \tag{32}$$

where

$$\Theta_\alpha = \sum_{\gamma=1}^{3} \left(\frac{\partial A_{\alpha\gamma}}{\partial x^\gamma} - \frac{1}{2} \frac{\partial A_{\gamma\gamma}}{\partial x^\alpha} \right). \tag{33}$$

The indices α, β take the values 1, 2, 3 and x^α are the cartesian coordinates in three-dimensional Euclidean space. Equation (33) contains six coupled partial differential equations with six unknowns, the $A_{\alpha\beta}$ with $A_{\alpha\beta} = A_{\beta\alpha}$. The adoption of a specific gauge condition for the metric tensor makes the right-hand side of (33) known and reduces the problem to that of equation (2). However, the study of the (physically acceptable) solutions of (33) remains an open problem and any progress will clarify the effect of gauge conditions on results and formulae connected with gravitational radiation.

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